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ASYMPTOTIC ANALYSIS OF THREE-DIMENSIONAL DYNAMIC EQUATIONS FOR THIN TWO-LAYER ELASTIC PLATES*

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In accordance with the method described in /1-3/, a derivation of two-dimensional equations of motion is given for a thin two-layer (non-symmetric) elastic plate. The mean values of the bending stiffness, the density, and Poisson's ratio are found, and the position of the middle plane is determined. In the coordinate system attached to this plane, the system of equations is separated into quasistatic equations for the longitudinal motion and a dynamic equation (of the ordinary kind) for the transverse component of the displacement. Unlike /1-3/, only one characteristic dimension in the longitudinal direction is introduced, which turns out to be sufficient and simplifies the analysis. Formulae of the complete field of stresses are provided. Stresses, which are of secondary importance for homogeneous plates, may be essential when the strength of the joint of the layers is considered.

1. Formulation of the problem. We shall consider a two-layer occupying a domain that is bounded or unbounded (in one or both directions). We denote by h_i, ρ_i, E_i, and ν_i the thickness, the density of the material and the elastic characteristics of the upper layer $(i=1)$ and lower layer $(i=2)$. We choose an orthogonal system of coordinates as shown in the figure. The xy-plane is parallel to the plane of the plate and the values $z=z_0$, z_1, z_2 determine the plane of complete contact of the layers and the face planes of the plate. On these boundaries we impose the following conditions:

$$\begin{gathered}\tau_{\alpha z}^{(i)}=\tau_{\alpha *}^{(i)}(\xi,\eta,\tau),\quad z=z_i\quad (\alpha=\xi,\eta,\zeta)\\ \tau_{\alpha z}^{(1)}=\tau_{\alpha z}^{(2)},\quad \mathbf{V}^{(1)}=\mathbf{V}^{(2)},\quad z=z_0\end{gathered}\tag{1.1}$$

where $\tau_{\alpha\beta}$ and $\mathbf{V}=(v_\xi, v_\eta, v_\zeta)$ are the dimensionless components of the stress tensor and the displacement vector, and $\tau_{\alpha *}^{(i)}$ are given fairly smooth functions of the longitudinal coordinates and time τ. We use different normalization of the functions and different scale extension for different directions:

$$\begin{gathered}\sigma_{\alpha\beta}=E_i\tau_{\alpha\beta},\quad \mathbf{u}=h\mathbf{V},\quad 2h=h_1+h_2\\ (x,y)=l(\xi,\eta),\ z=h\zeta,\ t=t_0\tau,\ \varepsilon=h/l\end{gathered}$$

Here l is the least characteristic linear dimension of the pattern of deformation in the longitudinal direction, and t_0 is the characteristic time defined as follows:

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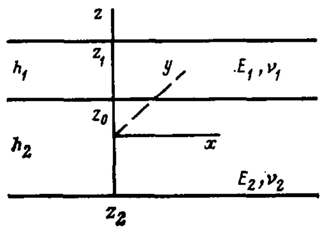
$$t_0 = \varepsilon^{\gamma-1} l c_1^{-1}, \quad c_1 = \sqrt{E_1/\rho_1}$$

where the parameter γ characterizes the variability of the state of stresses and displacements in time (below we consider the approximation $\gamma = 0$).



The quantity l has, especially in dynamics, a somewhat conditional character, and often can only be estimated a posteriori. Nevertheless, it is necessary to introduce it in the asymptotic analysis since the accuracy of the asymptotic equations can be defined only by means of the order of magnitude of the ratio h/l. Besides, unlike /1-3/, the least characteristic dimension of the plate in longitudinal directions L is considered below only as the upper bound of $l\ (l \leqslant L)$ and does not take any part in the analysis.

The three-dimensional dynamic equations of the theory of elasticity in dimensionless variables have the form (the index $i = 1, 2$ is sometimes omitted)

$$\frac{\partial^2 v_\beta}{\partial \zeta^2} + A_\beta - \varepsilon^{4-2\gamma} 2(1+\nu) C_i \frac{\partial^2 v_\beta}{\partial \tau^2} = 0 \quad (\beta = \xi, \eta) \tag{1.2}$$

$$\frac{\partial^2 v_\zeta}{\partial \zeta^2} + A_\zeta - \varepsilon^{4-2\gamma} \frac{(1-2\nu)(1+\nu)}{1-\nu} C_i \frac{\partial^2 v_\zeta}{\partial \tau^2} = 0, \quad C_i = \frac{\rho_i E_1}{E_i \rho_1}$$

$$A_\beta = \frac{\varepsilon}{1-2\nu} \frac{\partial^2 v_\zeta}{\partial \beta \partial \zeta} + \frac{\varepsilon^2}{1-2\nu} \frac{\partial}{\partial \beta} \operatorname{div} \mathbf{v} + \varepsilon^2 \Delta v_\beta$$

$$A_\zeta = \frac{\varepsilon}{2(1-\nu)} \frac{\partial}{\partial \zeta} \operatorname{div} \mathbf{v} + \frac{1-2\nu}{2(1-\nu)} \varepsilon^2 \Delta v_\zeta$$

$$\mathbf{v} = (v_\xi, v_\eta), \quad \Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2, \quad \operatorname{div} \mathbf{v} = \partial v_\xi/\partial\xi + \partial v_\eta/\partial\eta$$

$$\tau_\beta = \lambda_* \frac{\partial v_\zeta}{\partial \zeta} + \lambda_* \varepsilon \operatorname{div} \mathbf{v} + 2\mu_* \varepsilon \frac{\partial v_\beta}{\partial \beta}, \quad \tau_\zeta = (\lambda_* + 2\mu_*) \frac{\partial v_\zeta}{\partial \zeta} + \lambda_* \varepsilon \operatorname{div} \mathbf{v}$$

$$\tau_{\xi\eta} = \mu_* \varepsilon \left(\frac{\partial v_\xi}{\partial \eta} + \frac{\partial v_\eta}{\partial \xi} \right), \quad \tau_{\beta\xi} = \mu_* \left(\frac{\partial v_\beta}{\partial \zeta} + \varepsilon \frac{\partial v_\zeta}{\partial \beta} \right)$$

$$\lambda_* = \nu [(1-2\nu)(1+\nu)]^{-1}, \quad \mu_* = {}^1/_2 (1+\nu)^{-1}$$

To solve system of Eqs.(1.2), we shall use expansions of the functions being sought in asymptotic series in a small parameter ε /1-3/ (the summation index varies from $s = 0$ to $s = \infty$)

$$v_\beta = \varepsilon^{\varkappa+1} \sum \varepsilon^s v_\beta^{(s)}, \quad v_\zeta = \varepsilon^{\varkappa} \sum \varepsilon^s v_\zeta^{(s)}$$
$$\tau_\alpha = \varepsilon^{\varkappa} \sum \varepsilon^s \tau_\alpha^{(s)}, \quad \tau_{\xi\eta} = \varepsilon^{\varkappa+2} \sum \varepsilon^s \tau_{\xi\eta}^{(s)}, \quad \tau_{\beta\zeta} = \varepsilon^{\varkappa+1} \sum \varepsilon^s \tau_{\beta\zeta}^{(s)} \tag{1.3}$$

Eqs.(1.2) after substituting (1.3) can be integrated with respect to ζ, which generates representations of the functions $v_\beta^{(s)}, \ldots$ as partial sums of the series in powers of ζ:

$$v_\beta^{(s)} = \sum_{k=0}^{K+1} \zeta^k v_{\beta k}^{(s)}, \quad v_\zeta^{(s)} = \sum_{k=0}^{K} \zeta^k v_{\zeta k}^{(s)}$$
$$\tau_{\xi\eta}^{(s)} = \sum_{k=0}^{K+1} \zeta^k \tau_{\xi\eta k}^{(s)}, \quad \tau_{\beta\zeta}^{(s)} = \sum_{k=0}^{K} \zeta^k \tau_{\beta\xi k}^{(s)}, \quad \tau_\alpha^{(s)} = \sum_{k=0}^{K-1} \zeta^k \tau_{\alpha k}^{(s)} \tag{1.4}$$

where $K = 2m$ if $s = 2m$ or $s = 2m+1$. The index α takes the values ξ, η, and ζ, and the index $\beta = \xi, \eta$.

From Eqs.(1.2) we obtain recursion relations for the components $v_k^{(s)}$, which enables us to determine the components in terms of the quantities known from the foregoing approximations:

$$(k+2)(k+1) v_{\beta k+2}^{(s)} + \frac{k+1}{1-2\nu} \frac{\partial}{\partial\beta} v_{\zeta k+1}^{(s)} + \frac{1}{1-2\nu} \frac{\partial}{\partial\beta} \operatorname{div} \mathbf{v}_k^{(s-2)} + \tag{1.5}$$
$$\Delta v_{\beta k}^{(s-2)} - 2(1+\nu) C_i \frac{\partial^2 v_{\beta k}^{(s-4+2\gamma)}}{\partial \tau^2} = 0 \quad (k = 0, 1, \ldots, K-1)$$
$$(k+2)(k+1) v_{\zeta k+2}^{(s)} + \frac{k+1}{2(1-\nu)} \operatorname{div} \mathbf{v}_{k+1}^{(s-2)} + \frac{1-2\nu}{2(1-\nu)} \Delta v_{\zeta k}^{(s-2)} -$$
$$\frac{(1-2\nu)(1+\nu)}{1-\nu} C_i \frac{\partial^2 v_{\zeta k}^{(s-4+2\gamma)}}{\partial \tau^2} = 0 \quad (k = 0, 1, \ldots, K-2)$$

Moreover, we have the equations expressing the connection between the components of the stresses and the coordinates of the components of the displacements obtained as a result of substituting expressions (1.3) and (1.4) into Hook's law

$$\tau_{\beta k}^{(s)} = (k+1)\lambda_* v_{\zeta k+1}^{(s)} + 2\mu_* \frac{\partial v_{\beta k}^{(s-2)}}{\partial \beta} + \lambda_* \operatorname{div} \mathbf{v}_k^{(s-2)} \tag{1.6}$$

$$\tau_{\zeta k}^{(s)} = (k+1)(\lambda_* + 2\mu_*) v_{\zeta k+1}^{(s)} + \lambda_* \operatorname{div} \mathbf{v}_k^{(s-2)}$$

$$\tau_{\xi\eta k}^{(s)} = \mu_* \left(\frac{\partial v_{\xi k}}{\partial \eta} + \frac{\partial v_{\eta k}}{\partial \xi}\right)^{(s)} \quad \tau_{\beta\zeta k}^{(s)} = \mu_* \left((k+1) v_{\beta k+1} \frac{\partial v_{\zeta k}}{\partial \beta}\right)^{(s)}$$

2. ***The case*** $s=0, 1, K=0$. Under the assumption $\gamma < 2$, it follows from the system of Eqs.(1.2) written for the s-components of the functions without expanding them in the ζ-coordinate that

$$v_\zeta^{(s)} = v_{\zeta 0}^{(s)} + \zeta v_{\zeta 1}^{(s)}, \quad \tau_\beta^{(s)} = \lambda_* v_{\zeta 1}^{(s)}, \quad \tau_\zeta = (\lambda_* + 2\mu_*) v_{\zeta 1}^{(s)} \tag{2.1}$$

Moreover, considering the possibility that arbitrary boundary conditions (1.1) can be satisfied, we obtain $\tau_\zeta^{(s)} = v_{\zeta 1}^{(s)} = 0$ in the same way as in /1-3/. This means that the quantities satisfy the following system of relations for $s = 0, 1$:

$$v_{\zeta 1}^{(s)} = \tau_\alpha^{(s)} = \tau_{\beta\zeta}^{(s)} \equiv 0, \quad v_\zeta^{(s)} = v_{\zeta 0}^{(s)}, \quad v_\beta^{(s)} = v_{\beta 0}^{(s)} + \zeta v_{\beta 1}^{(s)} \tag{2.2}$$

$$v_{\beta 1}^{(s)} + \frac{\partial v_{\zeta 0}^{(s)}}{\partial \beta} = 0, \quad \tau_{\xi\eta}^{(s)} = \mu_* \left(\frac{\partial v_\xi}{\partial \eta} + \frac{\partial v_\eta}{\partial \xi}\right)^{(s)}$$

Moreover, from geometrical conditions (1.1) for the join of two layers and from equalities (2.2) we infer that

$$v_{\alpha 0}^{(1,s)} = v_{\alpha 0}^{(2,s)}, \quad v_{\beta 1}^{(1,s)} = v_{\beta 1}^{(2,s)} \tag{2.3}$$

where the digit in each superscript means that the given quantity refers to the 1st or the 2nd layer. Consequently, equalities (2.3) enable us to omit completely the first superscript of each of the above functions.

3. ***Quasistatic equations for the longitudinal components of the displacement (the case*** $s=2, 3, K=2$). We shall dwell on the question of finding the value of γ under the assumption that the ratios of the moduli of elasticity and of the densities of the layers do not introduce any new small or large parameters into the problem (otherwise, the analysis below would have to be modified).

For $\gamma < 0$, we obtain quasistatic equations for all components of the displacement. The equations hold if the external conditions change very slowly. The choice $\gamma > 0$ involves inertial terms that appear in the basic equations for both the longitudinal and the transverse components of the displacement. For the inertial terms to become significant in the equations for the longitudinal motion of the plate, the characteristic time should be commensurate with the time needed to traverse the length L of the plate. If $t_0 \ll Lc_1^{-1} = t_1$, it does not matter whether we take the additional terms into account in the equations for the longitudinal motion (in practice, it is sufficient that $2t_0 < t_1$).

In what follows we shall consider the intermediate case $\gamma = 0$. The physical meaning of this equality is that the characteristic time t_0 of the variablility of the state of stress and deformation is approximately ε^{-1}-fold greater than the time it takes the elastic wave to traverse the distance l. For harmonic oscillations of period T and of characteristic wave-length in the longitudinal direction $\lambda \sim 2l$, the condition $\gamma = 0$ can be rewritten as a restriction for the frequency $\omega = 2\pi/T \sim 4\pi h c_1 \lambda^{-2}$.

For the stress normal to the plane of the plate, we have $\tau_\zeta^{(s)} = \zeta\tau_{\zeta 1}^{(s)} + \tau_{\zeta 0}^{(s)}$. This arbitrary behaviour of the function $\tau_\zeta^{(s)}$ with respect to the ζ-coordinate is insufficient for arbitrary conditions on the face surfaces to be satisfied in the given order with respect to s. By analogy with the cases of a homogeneous plate /1-3/ and a symmetric three-layer plate /2/, it is required that

$$\tau_\zeta^{(s)} \equiv 0, \quad s = 0, 1, 2, 3 \tag{3.1}$$

should be set for the plates, which means that this stress is asymptotically small compared with the other stresses. The condition $\tau_{\beta\zeta}^{(s)} = 0\ (s = 0, 1)$ obtained earlier complies with the Kirchhoff-Love hypothesis. The relations below follow from conditions (3.1) and Eqs. (1.5):

$$v_{\beta 1}^{(s-2)} + \frac{\partial v_{\zeta 0}^{(s-2)}}{\partial \beta} = 0 \Rightarrow \operatorname{div} \mathbf{v}_1^{(s-2)} = -\Delta v_{\zeta 0}^{(s-2)} \tag{3.2}$$

$$v_{\zeta 1}^{(s)} + \frac{\nu}{1-\nu} \operatorname{div} \mathbf{v}_0^{(s-2)} = 0, \quad 2v_{\zeta 2}^{(s)} = \frac{\nu}{1-\nu} \Delta v_{\zeta 0}^{(s-2)} = -\frac{\nu}{1-\nu} \operatorname{div} \mathbf{v}_1^{(s-2)}$$

$$2v_{\beta 2}^{(s)} + \frac{\partial v_{\zeta 1}^{(s)}}{\partial \beta} + \frac{1+\nu}{1-\nu} \frac{\partial}{\partial \beta} \operatorname{div} \mathbf{v}_0^{(s-2)} + \Delta v_{\beta 0}^{(s-2)} = 0$$

$$3v_{\beta 3}^{(s)} + \frac{\partial v_{\zeta 2}^{(s)}}{\partial \beta} - \frac{1}{1-\nu} \frac{\partial}{\partial \beta} \Delta v_{\zeta 0}^{(s-2)} = 0 \quad (s = 2, 3)$$

For $s = 2, 3$, we rewrite the three conditions for the stress $\tau_{\beta\zeta}$ on the face planes and on the plane of contact in terms of $\tau_{\beta\zeta k}^{(s)}$, eliminating $\tau_{\beta\zeta 0}^{(1,s)}$ and $\tau_{\beta\zeta 0}^{(2,s)}$ from these equations. We obtain the relation

$$\sum_{i=1}^{2} (-1)^i E_i \{\delta_s^3 \tau_{\beta *}^{(i)} + (\zeta_0 - \zeta_i) \tau_{\beta\zeta 1}^{(i,s)} + (\zeta_0^2 - \zeta_i^2) \tau_{\beta\zeta 2}^{(i,s)}\} = 0 \tag{3.3}$$

$$\delta_s^s = 1, \quad \delta_s^m = 0, \quad m \neq s$$

From the basic Eqs.(1.6), taking into account the last two equations in (3.2), we have

$$\tau_{\beta\zeta 1}^{(s)} = -\frac{1}{2(1-\nu)} \frac{\partial}{\partial \beta} \operatorname{div} \mathbf{v}_0^{(s-2)} - \frac{1}{2(1+\nu)} \Delta v_{\beta 0}^{(s-2)} \tag{3.4}$$

$$\tau_{\beta\zeta 2}^{(s)} = \frac{1}{2(1+\nu)} \left(3v_{\beta 3}^{(s)} + \frac{\partial v_{\zeta 2}^{(s)}}{\partial \beta}\right) = \frac{1}{2(1-\nu^2)} \frac{\partial}{\partial \beta} \Delta v_{\zeta 0}^{(s-2)}$$

Substituting these expressions into (3.3), we obtain the required equations for the longitudinal motion

$$\left(\frac{\zeta_1 - \zeta_0}{1-\nu_1} - e\frac{\zeta_2 - \zeta_0}{1-\nu_2}\right) \frac{\partial}{\partial \beta} \operatorname{div} \mathbf{v}_0^{(s)} + \left(\frac{\zeta_1 - \zeta_0}{1+\nu_1} - e\frac{\zeta_2 - \zeta_0}{1+\nu_2}\right) \Delta v_{\beta 0}^{(s)} + \tag{3.5}$$

$$\left(\frac{\zeta_0^2 - \zeta_1^2}{1-\nu_1^2} - e\frac{\zeta_0^2 - \zeta_2^2}{1-\nu_2^2}\right) \frac{\partial}{\partial \beta} \Delta v_{\zeta 0}^{(s)} = -2\delta_s^1 (\tau_{\beta *}^{(1)} - e\tau_{\beta *}^{(2)}) \left(e = \frac{E_2}{E_1}, \quad s = 0, 1\right)$$

The equations for the transverse motion will be obtained in an analysis of the case $s = 4, 5$. To complete the investigations for $s = 2, 3$, we give the formulae for the stresses with the greatest order of magnitude with respect to ε:

$$\tau_{\beta\zeta}^{(s)} = \delta_s^3 \tau_{\beta *}^{(i)} + (\zeta - \zeta_i) \tau_{\beta\zeta 1}^{(s)} + (\zeta^2 - \zeta_i^2) \tau_{\beta\zeta 2}^{(s)}, \quad \tau_\beta^{(s)} = \tau_{\beta 0}^{(s)} + \zeta \tau_{\beta 1}^{(s)} \tag{3.6}$$

$$\tau_{\beta 0}^{(s)} = \frac{\nu}{1-\nu^2} \operatorname{div} \mathbf{v}_0^{(s-2)} + \frac{1}{1+\nu} \frac{\partial v_{\beta 0}^{(s-2)}}{\partial \beta}, \quad \tau_{\beta 1}^{(s)} = -\frac{\nu}{1-\nu^2} \Delta v_{\zeta 0}^{(s-2)} - \frac{1}{1+\nu} \frac{\partial^2 v_{\zeta 0}^{(s-2)}}{\partial \beta^2}$$

where the functions $\tau_{\zeta\beta k}^{(s)}$ are expressed in terms of $v_{\alpha 0}^{(s-2)}$ by formulae (3.4). The quantities $v_\alpha^{(s)}$, $\tau_{\xi\eta}^{(s)}$ $(s = 2, 3)$ are higher approximations compared with the displacements $v_\alpha^{(s-2)}$ and the stress $\tau_{\xi\eta}^{(s-2)}$ and are determined in the following step.

4. Equations for the bending motion of the plate (the case $s = 4, 5$). Since $\tau_{\beta\zeta}^{(s)} = 0$ for $s < 4$, the choice of the value $\varkappa = -4$ follows from the condition that the surface load is independent of the dimension h and the next term of the expansion of τ_ζ in a series in ε does not vanish. We eliminate the function $\tau_{\zeta 0}^{(1,s)}$ and $\tau_{\zeta 0}^{(2,s)}$ from the conditions for the stress τ_ζ on the planes $z = z_j$ $(j = 0, 1, 2)$:

$$\sum_{i=1}^{2} (-1)^i E_i \left[\sum_{k=1}^{3} (\zeta_0^k - \zeta_i^k) \tau_{\zeta k}^{(i,s)} + \delta_s^4 \tau_{\zeta *}^{(i)}\right] = 0 \tag{4.1}$$

We express the components $\tau_{\zeta k}^{(s)}$ $(k = 1, 2, 3)$ appearing in (4.1) in terms of $v_{\zeta 0}^{(s-4)}$. First, by using the equations of motion for the ζ-components of the displacement, we eliminate $v_{\zeta k+1}^{(s)}$ from Hooke's law (1.6) for these quantities

$$\tau_{\zeta k}^{(s)} = \frac{C_i}{k}\frac{\partial^2 v_{\zeta k}^{(s-4)}}{\partial \tau^2} - \mu_* \operatorname{div} \mathbf{v}_k^{(s-2)} - \frac{\mu_*}{k}\Delta v_{\zeta k-1}^{(s-2)}$$

For $k = 2$, and 3, we obtain expressions for $\operatorname{div} \mathbf{v}_k^{(s-2)}$ from the first two of the equations of motion (1.5) by replacing $k+2$ by k and applying the divergence operator to both sides of each of the equations:

$$\operatorname{div} \mathbf{v}_k^{(s-2)} = (2-\nu)(k\nu)^{-1}\Delta v_{\zeta k-1}^{(s-2)}$$

where a consequnence of the second row of formulae (3.2) is taken into account. Using the equality $v_{\zeta k-1}^{(s-1)} = 0$ with $k = 2, 3$, we find that it follows from the foregoing formulae in conjunction with the second and third equalities in (3.2) that

$$\tau_{\zeta 2}^{(s)} = \frac{1}{2(1-\nu^2)}\Delta \operatorname{div} \mathbf{v}_0^{(s-4)}, \quad \tau_{\zeta 3}^{(s)} = -\frac{1}{6(1-\nu^2)}\Delta\Delta v_{\zeta 0}^{(s-4)} \tag{4.2}$$

To transform the expression for the component $\tau_{\zeta 1}^{(s)}$, we use condition (1.1) for the stress $\tau_{\beta\zeta}$ and equalities (3.4) and (3.6):

$$\tau_{\zeta 1}^{(s)} = \left\{C_i \frac{\partial^2 v_{\zeta 0}}{\partial \tau^2} - \delta_s^5 \operatorname{div} \boldsymbol{\tau}_*^{(i)} - \frac{\zeta_i}{1-\nu_i^2}\Delta \operatorname{div} \mathbf{v}_0 + \frac{\zeta_i^2}{2(1-\nu_i^2)}\Delta\Delta v_{\zeta 0}\right\}^{(s-4)}$$

We substitute into (4.1) the expressions we found for the quantities $\tau_{\zeta k}^{(s)}$, and after some transformations we obtain the final expressions

$$d_0 \Delta\Delta v_{\zeta 0}^{(s)} + d_1 \frac{\partial^2 v_{\zeta 0}^{(s)}}{\partial \tau^2} = \delta_s^0 (\tau_{\zeta *}^{(1)} - e\tau_{\zeta *}^{(2)}) + \delta_s^1 h^{-1} \operatorname{div}(h_1 \boldsymbol{\tau}_*^{(1)} + eh_2 \boldsymbol{\tau}_*^{(2)} + p_0 h \Delta v_0^{(s)}) \tag{4.3}$$

$$d_0 = \frac{h_1^2}{h^2}\frac{2\zeta_1+\zeta_0}{6(1-\nu_1^2)} - \frac{h_2^2}{h^2}\frac{e(2\zeta_2+\zeta_0)}{6(1-\nu_2^2)}, \quad d_1 = \frac{h_1\rho_1 + h_2\rho_2}{h\rho_1}$$

$$p_0 = \frac{h_1^2}{2h^2(1-\nu_1^2)} - \frac{eh_2^2}{2h^2(1-\nu_2^2)} \quad (s = 0, 1)$$

If the coordinate system is fixed so that the coefficient associated with $\Delta v_{\zeta 0}^{(s)}/\partial\beta$ in (3.5) vanishes, then the system of Eqs.(3.5) and (4.3) can be separated into a system of quasistatic equations for the longitudinal components of the displacements $v_{\beta 0}^{(s)}$ only and into dynamic equations for the transverse motion in the same way as in the case of a homogeneous plate /3/. Using this condition one can uniquely determine the position of the middle plane. Namely, for the coordinate $\zeta_0 = z_0/h$, we have

$$\zeta_0 = \frac{\varkappa_0 h_2^2 - h_1^2}{2h(\varkappa_0 h_2 + h_1)}, \quad \zeta_i = \zeta_0 - \frac{h_i(-1)^i}{h}, \quad \varkappa_0 = e\frac{1-\nu_1^2}{1-\nu_2^2} \tag{4.4}$$

The relations below follow from Eqs.(3.5):

$$\operatorname{div}\Delta \mathbf{v}_0^{(s)} = \Delta \operatorname{div} \mathbf{v}_0^{(s)} = -q_0 \delta_s^1 \operatorname{div}(\boldsymbol{\tau}_*^{(1)} - e\boldsymbol{\tau}_*^{(2)}), \quad q_0 = \left[\frac{h_1 h^{-1}}{1-\nu_1^2} + \frac{eh_2 h^{-1}}{1-\nu_2^2}\right]^{-1} \tag{4.5}$$

The operators Δ and div commute in Cartesian coordinates. Substitution of the expressions for $\operatorname{div} \Delta \mathbf{v}_0^{(s)}$ into Eq.(4.3) leads to independent equations for the main part of the transverse displacement.

Eqs.(3.5) and (4.3) are obtained by construction to an accuracy $O(\varepsilon^2)$ from the three-dimensional equations of the theory of elasticity. The classical boundary conditions of the theory of plates constitute natural boundary conditions for these equations. The inaccuracy of the solutions introduced by integral conditions on the butt-ends depends, generally speaking, on the choice of these conditions (it is possible to state accuracy boundary conditions /4/), but in the cases considered earlier in /1-4/ it does not exceed $O(\varepsilon)$ anywhere in the domain of definition except for the boundary layer. For a multilayer plane this question calls for a separate treatment.

Since the problem is separated, it is advisable to state formulae describing the reactions caused separately by the normal load and the tangential load. In this connection, we make an implicit assumption about the presence of homogeneous boundary conditions for the displacements and their derivatives on the butt-ends of the plate.

5. Complete systems of equations describing the fields of displacements and stresses in the two-dimensional theory. When only the normal load is applied $(\tau_{\beta *}^{(i)} = 0)$, the coordinates $v_{\xi 0}^{(s)}$, $v_{\eta 0}^{(s)} v_{\zeta 0}^{(1)}$ $(s = 0, 1)$ of the displacement will vanish. The single equation for the transverse displacement $u_z = h v_{\zeta 0}^{(0)}$ and the expressions for the stresses and longitudinal displacements in terms of the transverse displacement written using dimensional variables have the following form (see formulae (1.3), (2.1), (2.2), (3.4), (3.6), (4.2) and (4.3)):

$$D_* \Delta\Delta u_z + 2h\rho_* \frac{\partial^2 u_z}{\partial t^2} = \sigma_{z*}^{(1)} - \sigma_{z*}^{(2)} \tag{5.1}$$

$$D_* = h^3 d_0 E_1, \quad \rho_* = (2h)^{-1}(\rho_1 h_1 + \rho_2 h_2)$$

$$u_\rho = -z\frac{\partial u_z}{\partial \rho}, \quad \sigma_\rho^{(i)} = -\frac{zE_i}{1-\nu_i^2}\left[\nu_i \Delta u_z + (1-\nu_i)\frac{\partial^2 u_z}{\partial \rho^2}\right] \quad (\rho = x, y)$$

$$\sigma_{xy}^{(i)} = -\frac{zE_i}{1+\nu_i}\frac{\partial^2 u_z}{\partial x \partial y}, \quad \sigma_{\rho z}^{(i)} = E_i \frac{z^2 - z_i^2}{2(1-\nu_i^2)}\frac{\partial}{\partial \rho}\Delta u_z$$

$$\sigma_z^{(i)} = F_i E_i h c_1^{-2}\frac{\partial^2 u_z}{\partial t^2} + \frac{G_i E_i}{E_1}(\sigma_{z*}^{(2)} - \sigma_{z*}^{(1)}) + \sigma_{z*}^{(i)}$$

$$F_i = C_i \frac{z - z_i}{h} + d_1 G_i, \quad G_i = \frac{(z - z_i)^2 (z + 2z_i)}{6 d_0 h^3 (1 - \nu_i^2)}$$

$$M_x = -D_*\left(\frac{\partial^2 u_z}{\partial x^2} + \nu_* \frac{\partial^2 u_z}{\partial y^2}\right), \quad Q_{xz} = -D_* \frac{\partial}{\partial x}\Delta u_z \quad (x \leftrightarrow y)$$

$$M_{xy} = -D_*(1-\nu_*)\frac{\partial^2 u_z}{\partial x \partial y}$$

$$\nu_* D_* = \left(4 + 6\frac{z_0}{h_1}\right)\nu_1 D_1 + \left(4 - 6\frac{z_0}{h_2}\right)\nu_2 D_2, \quad D_i = \frac{h_i^3 E_i}{12(1-\nu_i^2)}$$

Here we also give expressions for the bending moments and shear forces. They are obtained by integrating the stresses using the usual integral representations for these quantities. The moments are evaluated with respect to the middle plane, whose position is determined by formula (4.4).

Thus, it is proved that all the relationships of the classical theory of bending are preserved in the coordinate system fixed according to formula (4.4) with the mean values of the bending stiffness D_*, the density ρ_*, and Poisson's ratio ν_* being used in the equation of motion and in the expressions for the moments and forces in (5.1). These parameters turn into those of a homogeneous plate if the appropriate passages to the limit are performed. All the remaining formulae for the stresses and longitudinal displacements (except for the formula for the stress σ_z) are identical with the classical formulae inside each of the layers. The bending stresses undergo a discontinuity on the boundary between the plates, while the other quantities are continuous on this boundary. The behaviour as $h \to 0$ is in agreement with the case of a homogeneous plate.

$$u_z \sim h^{-3}, \quad u_\rho, \sigma_\rho, \quad \sigma_{xy} \sim h^{-2}, \quad \sigma_{\rho z} \sim h^{-1}, \quad \sigma_z \sim h^0$$

The reaction of a two-layer plate caused by a tangential load is described in the main part by the following system of relations. All coordinates of the displacements with the index $s = 0$ vanish. From formulae (1.3), (1.4), (2.2), (3.5) (3.6) and (4.2)-(4.5), we obtain the equations for the remaining coordinates of displacements and the expressions for the stresses written in dimensional form

$$\Delta \mathbf{u}_0 + p\nabla \operatorname{div} \mathbf{u}_0 = 2q_+ E_1^{-1}(\boldsymbol{\sigma}_*^{(1)} - \boldsymbol{\sigma}_*^{(2)}), \quad \mathbf{u}_0 = h\mathbf{v}_0^{(1)} \tag{5.2}$$

$$D_* \Delta\Delta u_z + 2h\rho_* \frac{\partial^2 u_z}{\partial t^2} = hp_1 \operatorname{div} \boldsymbol{\sigma}_*^{(1)} - hp_2 \operatorname{div} \boldsymbol{\sigma}_*^{(2)}$$

$$u_\rho = u_{\rho 0} - z\frac{\partial u_z}{\partial \rho} \quad (\rho = x, y)$$

$$\sigma_\rho^{(i)} = \frac{E_i}{1-\nu_i^2}\left[\nu_i \operatorname{div} \mathbf{u}_0 + (1-\nu_i)\frac{\partial u_{\rho 0}}{\partial \rho}\right] - \frac{zE_i}{1-\nu_i^2}\left[(\nu_i \Delta u_z + (1-\nu_i)\frac{\partial^2 u_z}{\partial \rho^2}\right]$$

$$\sigma_{\rho z}^{(i)} = \sigma_{\rho *}^{(i)} + \frac{z_i - z}{2(1-\nu_i)} E_i \left(\frac{\partial}{\partial \rho}\operatorname{div} \mathbf{u}_0 + \frac{1-\nu_i}{1+\nu_i}\Delta u_{\rho 0}\right) + \frac{z^2 - z_i^2}{2(1-\nu_i^2)} E_i \frac{\partial}{\partial \rho}\Delta u_z$$

$$\sigma_{xy}^{(i)} = \frac{E_i}{2(1+\nu_i)}\left(\frac{\partial u_{x0}}{\partial y} + \frac{\partial u_{y0}}{\partial x} - 2z\frac{\partial^2 u_z}{\partial x \partial y}\right)$$

$$\sigma_z^{(i)} = hc_1^{-2}F_iE_i\frac{\partial^2 u_z}{\partial t^2} + \operatorname{div}\left\{\frac{q_0E_i(z-z_i)^2}{2hE_1(1-\nu_i^2)}(\sigma_*^{(1)} - \sigma_*^{(2)}) - \right.$$

$$\left.(z-z_i)\sigma_*^{(i)} + hG_i\frac{E_i}{E_1}(p_1\sigma_*^{(1)} - p_1\sigma_*^{(2)})\right\}$$

$$q_\pm = \left(\frac{h_1}{1\pm\nu_1} + \frac{eh_2}{1\pm\nu_2}\right)^{-1}, \quad p = \frac{q_+}{q_-}, \quad p_i = p_0q_0 - (-1)^i\frac{h_i}{h}$$

where $\sigma_*^{(i)}$ is a vector representing tangential forces imposed on the face sides.

Eqs.(5.2) have the same form as the analogous equations for the longitudinal motion of a plate given in /3/. If $\nu_1 = \nu_2$, $E_1 = E_2$ or $h_i = 0$ ($i = 1$ or 2), the equations are completely identical. Variable tangential forces result in a non-vanishing longitudinal displacement, which obeys the second equation in (5.2). This equation differs from the dynamic equation in (5.1) by a free term only. For a constant tangential load, $u_z = 0$ and the system of relationships (5.2) becomes much simpler. The order of all quantities with respect to h as $h \to 0$ is less by one than in the case of bending.

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